Game-theoretic Brownian motion

Vladimir Vovk
vovk@cs.rhul.ac.uk
http://vovk.net

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Abstract

This paper suggests a perfect-information game, along the lines of Lévy's characterization of Brownian motion, that formalizes the process of Brownian motion in game-theoretic probability. This is perhaps the simplest situation where probability emerges in a non-stochastic environment.

1 Introduction

This paper is part of the recent revival of interest (see, e.g., [4, 14, 15, 10, 6, 9]) in game-theoretic probability. It further develops the game-theoretic approach to continuous-time processes along the lines of the papers [16, 18, 19]. Unlike those papers, which only demonstrate the emergence of randomness-type properties in a continuous trading protocol, this paper gives an example where a full-fledged probability measure emerges (although in a significantly more restrictive protocol). Only the very simple case of "game-theoretic Brownian motion" is considered, but we can expect that probabilities will also emerge in less restrictive protocols.

The words such as "positive", "negative", "before", and "after" will be understood in the wide sense of $\geq$ or $\leq$, as appropriate; when necessary, we will add the qualifier "strictly".

The latest version of this working paper can be downloaded from the web site `http://probabilityandfinance.com`.

2 Upper and lower probability

We consider a perfect-information game between two players, Reality and Sceptic. Reality chooses a continuous function $\omega : [0, \infty) \to \mathbb{R}$ with $\omega(0) = 0$, but before she announces her choice Sceptic chooses his strategy of trading in two securities, one with the price process $\omega(t)$, $t \in [0, \infty)$, and the other with the price process $\omega^2(t) - t$, $t \in [0, \infty)$. This game, which we call the *Lévy game* (after [11], Theorem 18.6), is formalized as follows.

Let Ω be the set of all continuous functions $\omega : [0,\infty) \to \mathbb{R}$ with $\omega(0) = 0$. For each $t \in [0,\infty)$, $\mathcal{F}_t$ is defined to be the σ-algebra generated by the functions $\omega \in \Omega \mapsto \omega(s)$, $s \in [0,t]$, and $\mathcal{F}_\infty := \vee_t \mathcal{F}_t$; we will often write $\mathcal{F}$ for $\mathcal{F}_\infty$. A *process* S is a family of functions $S_t : \Omega \to [-\infty,\infty]$, $t \in [0,\infty)$, each S_t being $\mathcal{F}_t$-measurable; we only consider processes with lower continuous (often continuous) *sample paths* $t \mapsto S_t(\omega)$. An *event* is an element of the σ-algebra $\mathcal{F}$. Stopping times $\tau : \Omega \to [0,\infty]$ w.r. to the filtration $(\mathcal{F}_t)$ and the corresponding σ-algebras $\mathcal{F}_\tau$ are defined as usual. We simplify $\omega(\tau(\omega))$ and $S_{\tau(\omega)}(\omega)$ to $\omega(\tau)$ and $S_\tau(\omega)$, respectively; the argument ω will often be omitted in other cases as well.

The class of allowed strategies for Sceptic is defined in two steps. An *elementary betting strategy* G consists of an increasing sequence of stopping times $\tau_1 \le \tau_2 \le \cdots$ and, for each $n = 1, 2, \ldots$, a pair of bounded $\mathcal{F}_{\tau_n}$-measurable functions, M_n and V_n. It is required that, for any $\omega \in \Omega$, $\lim_{n\to\infty} \tau_n(\omega) = \infty$. To such G and an *initial capital* $c \in \mathbb{R}$ corresponds the *elementary capital process*

$$\begin{aligned}\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} \Bigg(& M_n(\omega)\big(\omega(\tau_{n+1}\wedge t) - \omega(\tau_n \wedge t)\big) \\ & + V_n(\omega)\Big(\big(\omega^2(\tau_{n+1}\wedge t) - (\tau_{n+1}\wedge t)\big) - \big(\omega^2(\tau_n \wedge t) - (\tau_n \wedge t)\big)\Big)\Bigg), \\ & \qquad t \in [0,\infty) \qquad (1)\end{aligned}$$

(with the zero terms in the sum ignored). The numbers $M_n(\omega)$ and $V_n(\omega)$ will be called Sceptic's *stakes* (on $\omega(t)$ and $\omega^2(t) - t$, respectively) chosen at time τ_n, and $\mathcal{K}_t^{G,c}(\omega)$ will sometimes be referred to as Sceptic's capital at time t; we may also say that Sceptic *bets* $M_n(\omega)$ on $\omega(t)$ and $V_n(\omega)$ on $\omega^2(t) - t$ at time τ_n. (We are following standard probability textbooks, such as [20], Chapter 10, in using gambling rather than financial terminology.)

A *positive capital process* is any process S that can be represented in the form

$$S_t(\omega) := \sum_{n=1}^{\infty} \mathcal{K}_t^{G_n,c_n}(\omega), \qquad (2)$$

where the elementary capital processes $\mathcal{K}_t^{G_n,c_n}(\omega)$ are required to be positive, for all t and ω, and the positive series $\sum_{n=1}^{\infty} c_n$ is required to converge. The sum (2) is always positive but allowed to take value ∞. Since $\mathcal{K}_0^{G_n,c_n}(\omega) = c_n$ does not depend on ω, $S_0(\omega)$ also does not depend on ω and will often be abbreviated to S_0.

The *upper probability* of a set $E \subseteq \Omega$ is defined as

$$\overline{\mathbb{P}}(E) := \inf\big\{S_0 \bigm| \forall\omega\in\Omega : \liminf_{t\to\infty} S_t(\omega) \ge \mathbb{I}_E(\omega)\big\}, \qquad (3)$$

where S ranges over the positive capital processes and $\mathbb{I}_E$ stands for the indicator function of E. (The $\liminf_{t\to\infty}$ can be replaced by $\sup_{t\in[0,\infty)}$ in this definition: see [18], Lemma 1.) The *lower probability* of $E \subseteq \Omega$ is

$$\underline{\mathbb{P}}(E) := 1 - \overline{\mathbb{P}}(E^c),$$

where $E^c := \Omega \setminus E$ stands for the complement of E.

Remark. Our definition of a positive capital process corresponds to the intuitive picture where Sceptic divides his initial capital into a sequence of independent accounts, with a prudent (not risking bankruptcy) elementary betting strategy applied to each account. On the other hand, we could make the definition of upper probability more similar to the standard definition of expectation for positive random variables: a positive capital process could be equivalently defined as the limit of an increasing sequence of positive elementary capital processes with uniformly bounded initial capitals.

3 Emergence of the Wiener measure

It is obvious that upper probability is countably (in particular, finitely) subadditive:

Lemma 1. *For any sequence of subsets $E_1, E_2, \dots$ of Ω,*

$$\overline{\mathbb{P}}\left(\bigcup_{n=1}^{\infty} E_n\right) \le \sum_{n=1}^{\infty} \overline{\mathbb{P}}(E_n).$$

Therefore, $\overline{\mathbb{P}}$ is an outer measure in Carathéodory's sense. Recall that a set $A \subseteq \Omega$ is $\overline{\mathbb{P}}$-*measurable* if, for each $E \subseteq \Omega$,

$$\overline{\mathbb{P}}(E) = \overline{\mathbb{P}}(E \cap A) + \overline{\mathbb{P}}(E \cap A^c). \tag{4}$$

A standard result (see [3], Sections 1–11, or, e.g., [7], Theorem 2.1) shows that the family $\mathcal{A}$ of all $\overline{\mathbb{P}}$-measurable sets forms a σ-algebra and that the restriction of $\overline{\mathbb{P}}$ to $\mathcal{A}$ is a probability measure on $(\Omega, \mathcal{A})$.

Theorem 1. *Each event $A \in \mathcal{F}$ is $\overline{\mathbb{P}}$-measurable, and the restriction of $\overline{\mathbb{P}}$ to $\mathcal{F}$ coincides with the Wiener measure W on $(\Omega, \mathcal{F})$. In particular, $\overline{\mathbb{P}}(A) = \underline{\mathbb{P}}(A) = W(A)$ for each $A \in \mathcal{F}$.*

The rest of this paper is devoted to proving this result.

4 Statement in terms of expectation

A *capital process* is a process of the form $S - C$, where S is a positive capital process and C is a real constant. The *upper expectation* of a bounded functional $F : \Omega \to \mathbb{R}$ is

$$\overline{\mathbb{E}}(F) := \inf\left\{S_0 \;\middle|\; \forall \omega \in \Omega : \liminf_{t \to \infty} S_t(\omega) \ge F(\omega)\right\},$$

where S ranges over the capital processes. (This generalizes upper probability: $\overline{\mathbb{P}}(E) = \overline{\mathbb{E}}(\mathbb{I}_E)$ for all $E \subseteq \Omega$.) Theorem 1 will immediately follow from the following result:

Theorem 2. *If F is a bounded $\mathcal{F}$-measurable functional on Ω,*

$$\overline{\mathbb{E}}(F) = \int_{\Omega} F(\omega) W(d\omega). \tag{5}$$

Indeed, let us deduce Theorem 1 from Theorem 2. Let $A \in \mathcal{F}$. Since the inequality $\leq$ in (4) follows from Lemma 1, to show that $A \in \mathcal{A}$ we are only required to show

$$\overline{\mathbb{P}}(E \cap A) + \overline{\mathbb{P}}(E \cap A^c) \leq \overline{\mathbb{P}}(E) + \epsilon \tag{6}$$

for each $\epsilon > 0$. Fix such an ϵ.

Let S be a positive capital process such that $S_0 < \overline{\mathbb{P}}(E) + \epsilon$ and $\forall \omega \in \Omega$: $\liminf_{t\to\infty} S_t(\omega) \geq \mathbb{I}_E(\omega)$. Set $F(\omega) := 1 \wedge \liminf_{t\to\infty} S_t(\omega)$, so that F is a bounded (taking values in $[0,1]$) $\mathcal{F}$-measurable functional satisfying $\overline{\mathbb{E}}(F) < \overline{\mathbb{P}}(E) + \epsilon$, $\mathbb{I}_{E\cap A} \leq F\,\mathbb{I}_A$, and $\mathbb{I}_{E\cap A^c} \leq F\,\mathbb{I}_{A^c}$ (the last two inequalities follow from $\mathbb{I}_E \leq F$). Therefore, it suffices to prove

$$\overline{\mathbb{E}}(F\,\mathbb{I}_A) + \overline{\mathbb{E}}(F\,\mathbb{I}_{A^c}) \leq \overline{\mathbb{E}}(F).$$

This immediately follows from $F\,\mathbb{I}_A + F\,\mathbb{I}_{A^c} = F$ and Theorem 2.

The remaining statements of Theorem 1 are obvious corollaries of Theorem 2: for $A \in \mathcal{F}$,

$$\overline{\mathbb{P}}(A) = \overline{\mathbb{E}}(\mathbb{I}_A) = \int_{\Omega} \mathbb{I}_A \, dW = W(A),$$
$$\underline{\mathbb{P}}(A) = 1 - \overline{\mathbb{P}}(A^c) = 1 - W(A^c) = W(A).$$

5 Coherence and its application

In this and the following two sections we will prove some auxiliary results that will be needed in the proof of Theorem 2. This section's results, however, also have considerable substantive significance: they show that our definitions are "free of contradiction". (In fact, these definitions have been chosen to make this easy.)

The following result says that the Lévy game is *coherent*, in the sense that $\overline{\mathbb{P}}(\Omega) = 1$ (i.e., no positive capital process increases its value between time 0 and ∞ by more than a positive constant for all $\omega \in \Omega$).

Proposition 1. $\overline{\mathbb{P}}(\Omega) = 1$.

Proof. If ω is generated as sample path of (measure-theoretic) Brownian motion, any positive elementary capital process will be a positive continuous local martingale (since, by the optional sampling theorem, every partial sum in (1) will be a continuous martingale), and so it suffices to apply the maximal inequality for positive supermartingales to the partial sums corresponding to a given positive capital process. □

The *lower expectation* of a bounded functional $F : \Omega \to \mathbb{R}$ is defined as

$$\underline{\mathbb{E}}(F) := -\,\overline{\mathbb{E}}(-F).$$

Corollary 1. *For every bounded functional on Ω, $\underline{\mathbb{E}}(F) \le \overline{\mathbb{E}}(F)$.*

Proof. Suppose $\underline{\mathbb{E}}(F) > \overline{\mathbb{E}}(F)$ for some F; by the definition of $\underline{\mathbb{E}}$, this would mean that $\overline{\mathbb{E}}(F) + \overline{\mathbb{E}}(-F) < 0$. Since $\overline{\mathbb{E}}$ is finitely subadditive, this would imply $\overline{\mathbb{E}}(0) < 0$, which is equivalent to $\overline{\mathbb{P}}(\Omega) < 1$ and, therefore, impossible by Proposition 1. $\square$

Specializing Corollary 1 to the indicator functions of subsets of Ω, we obtain:

Corollary 2. *For every set $A \subseteq \Omega$, $\underline{\mathbb{P}}(A) \le \overline{\mathbb{P}}(A)$.*

In the special case where A is $\overline{\mathbb{P}}$-measurable (see (4)), Proposition 1 implies $\overline{\mathbb{P}}(A) = \underline{\mathbb{P}}(A)$.

The facts that we have just established allow us to simplify our goal: (5) will follow from

$$\overline{\mathbb{E}}(F) \le \int_\Omega F(\omega) W(d\omega). \tag{7}$$

Indeed, (7) implies

$$\underline{\mathbb{E}}(F) = -\,\overline{\mathbb{E}}(-F) \ge -\int_\Omega (-F(\omega)) W(d\omega) = \int_\Omega F(\omega) W(d\omega),$$

and so Corollary 1 implies

$$\overline{\mathbb{E}}(F) = \underline{\mathbb{E}}(F) = \int_\Omega F(\omega) W(d\omega).$$

6 Modified Lévy game

Theorem 2 (as well as Theorem 1) also holds, and will be slightly easier to prove, for the natural modification of the Lévy game in which Sceptic is allowed to bet on $\omega(t) - \omega(\tau)$ and $(\omega(t) - \omega(\tau))^2 - (t - \tau)$ at any time τ. Formally, we obtain the definition of upper probability in the modified Lévy game when we replace (1) by

$$\mathcal{K}_t^{G,c}(\omega) := c + \sum_{n=1}^{\infty} \Biggl(M_n(\omega) \bigl(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\bigr) \\ + V_n(\omega) \left(\bigl(\omega(\tau_{n+1} \wedge t) - \omega(\tau_n \wedge t)\bigr)^2 - \bigl((\tau_{n+1} \wedge t) - (\tau_n \wedge t)\bigr) \right) \Biggr);$$

we will say that the corresponding elementary betting strategy bets $M_n(\omega)$ (or stakes $M_n(\omega)$ units) on $\omega(t) - \omega(\tau_n)$ and bets $V_n(\omega)$ (or stakes $V_n(\omega)$ units) on $(\omega(t) - \omega(\tau_n))^2 - (t - \tau_n)$ at time τ_n. The rest of the definition is as before: positive capital processes are defined by (2) and upper probability is then defined by (3). The definitions of lower probability and upper and lower expectation also carry over to the modified Lévy game.

The Lévy game and modified Lévy game are very close, as can be seen from the identity

$$\begin{aligned} M_n(\omega)\bigl(\omega(\tau_{n+1}\wedge t)-\omega(\tau_n\wedge t)\bigr) \\ + V_n(\omega)\left(\bigl(\omega(\tau_{n+1}\wedge t)-\omega(\tau_n\wedge t)\bigr)^2-\bigl((\tau_{n+1}\wedge t)-(\tau_n\wedge t)\bigr)\right) \\ = \bigl(M_n(\omega)-2\omega(\tau_n\wedge t)V_n(\omega)\bigr)\bigl(\omega(\tau_{n+1}\wedge t)-\omega(\tau_n\wedge t)\bigr) \\ + V_n(\omega)\Bigl(\bigl(\omega^2(\tau_{n+1}\wedge t)-(\tau_{n+1}\wedge t)\bigr)-\bigl(\omega^2(\tau_n\wedge t)-(\tau_n\wedge t)\bigr)\Bigr). \end{aligned} \tag{8}$$

However, the absence of an upper bound on ω prevents us from asserting that the two games lead to the same notion of upper probability. (Remember that the stakes are required to be bounded, which is implicitly used in the proof of Proposition 1.) If there is a risk of confusion, we will write $\overline{\mathbb{P}}'$, $\underline{\mathbb{P}}'$, $\overline{\mathbb{E}}'$, and $\underline{\mathbb{E}}'$ instead of $\overline{\mathbb{P}}$, $\underline{\mathbb{P}}$, $\overline{\mathbb{E}}$, and $\underline{\mathbb{E}}$, respectively, for the modified Lévy game.

It is obvious that Lemma 1 continues to hold for the modified Lévy game. This is also true about Proposition 1 and Corollaries 1–2 (although this fact is not used in this paper outside this section). As already mentioned, Theorem 1 and Theorem 2 also hold for the modified Lévy game: we will prove (7) directly for this game, and the reduction to (7) depended on arguments of general nature, not involving the specifics of the Lévy game.

7 Tightness in the modified Lévy game

The set Ω is equipped with the standard metric

$$\rho(\omega_1,\omega_2) := \sum_{n=1}^{\infty} 2^{-n} \sup_{t\in[0,n]} \left(|\omega_1(t)-\omega_2(t)|\wedge 1\right), \tag{9}$$

which makes Ω a complete separable metric space with Borel σ-algebra $\mathcal{F}$. In this topology, $\underline{\mathbb{P}}'$ is tight:

Lemma 2. *For each $\alpha>0$ there exists a compact set $K\subseteq\Omega$ such that $\underline{\mathbb{P}}'(K)\ge 1-\alpha$.*

For $\omega\in\Omega$ and $T\in(0,\infty)$, the *modulus of continuity of ω on* $[0,T]$ is defined as

$$m_\delta^T(\omega) := \sup_{s,t\in[0,T]:|s-t|\le\delta} |\omega(s)-\omega(t)|, \quad \delta>0.$$

The proof of Lemma 2 will be based on the following result.

Lemma 3. *For each $\alpha>0$ and $T>0$,*

$$\underline{\mathbb{P}}'\left\{\forall\delta>0 : m_\delta^T \le 157\,\alpha^{-1/2}T^{3/8}\delta^{1/8}\right\} \ge 1-\alpha. \tag{10}$$

Of course, Theorem 1 and its counterpart for the modified Lévy game will imply much subtler results than Lemma 3 and its counterpart for the Lévy game, such as Lévy's modulus of continuity formula. It is interesting, however, that this section's results do not require that Sceptic should be allowed to bet on $\omega(t)-\omega(\tau)$ (i.e., he can achieve his goal even if he is required to always choose $M_n(\omega):=0$).

Proof of Lemma 3. We will use the method of [17], pp. 213–216. For each $n=1,2,\ldots$, divide the time interval $[0,T]$ into 2^n equal subintervals of length $2^{-n}T$. Fix, for a moment, an n, and set $\beta=\beta_n:=\left(2^{1/4}-1\right)2^{-n/4}\alpha$ (where $2^{1/4}-1$ is the normalizing constant ensuring that the β_n sum to α) and

$$\omega_i:=\omega(i2^{-n}T),\quad i=0,1,\ldots,2^n.$$

With lower probability at least $1-\beta/2$,

$$\sum_{i=1}^{2^n}(\omega_i-\omega_{i-1})^2\le 2T/\beta \tag{11}$$

(since there is a positive elementary capital process taking value $T+\sum_{i=1}^{j}(\omega_i-\omega_{i-1})^2-j2^{-n}T$ at time $j2^{-n}T$, $j=0,1,\ldots,2^n$, and this elementary capital process will make $2T/\beta$ at time T out of initial capital T if (11) fails to happen). For each $\omega\in\Omega$, define

$$J(\omega):=\left\{i=1,\ldots,2^n:|\omega_i-\omega_{i-1}|\ge\epsilon\right\},$$

where $\epsilon=\epsilon(\delta,n)$ will be chosen later. It is clear that $|J(\omega)|\le 2T/\beta\epsilon^2$ on the set (11). Consider the elementary betting strategy that bets 1 on $(\omega(t)-\omega(\tau))^2-(t-\tau)$ at each time $\tau\in[(i-1)2^{-n}T,i2^{-n}T]$ with $i\in J(\omega)$ when $|\omega(\tau)-\omega_{i-1}|=\epsilon$ for the first time during $[(i-1)2^{-n}T,i2^{-n}T]$ and gets rid of the stake at time $i2^{-n}T$. This strategy will make at least ϵ^2 out of $(2T/\beta\epsilon^2)2^{-n}T$ provided both event (11) and event

$$\exists i\in\{1,\ldots,2^n\}:|\omega_i-\omega_{i-1}|\ge 2\epsilon$$

happen. (And we can make the corresponding elementary capital process positive by allowing Sceptic to bet at most $2T/\beta\epsilon^2$ times.) This corresponds to making at least 1 out of $(2T/\beta\epsilon^4)2^{-n}T$. Solving the equation $(2T/\beta\epsilon^4)2^{-n}T=\beta/2$ gives $\epsilon=(4T^2 2^{-n}/\beta^2)^{1/4}$. Therefore,

$$\begin{aligned}\max_{i=1,\ldots,2^n}|\omega_i-\omega_{i-1}|&\le 2\epsilon=2(4T^2 2^{-n}/\beta^2)^{1/4}\\&=2^{3/2}\left(2^{1/4}-1\right)^{-1/2}\alpha^{-1/2}T^{1/2}2^{-n/8}\end{aligned} \tag{12}$$

with lower probability at least $1-\beta$. By the countable subadditivity of upper probability (Lemma 1), (12) holds for all $n=1,2,\ldots$ with lower probability at least $1-\sum_n\beta_n=1-\alpha$.

Intervals of the form $[(i-1)2^{-n}T, i2^{-n}T]$, for $n \in \{1,2,\ldots\}$ and $i \in \{1,2,3,\ldots,2^n\}$, will be called *dyadic*. Given an interval $[s,t]$ of length at most $\delta > 0$ in $[0,T]$, we can cover its interior (without covering any points in its complement) by adjacent dyadic intervals with disjoint interiors such that, for some $m \in \{1,2,\ldots\}$: there are between one and two dyadic intervals of length $2^{-m}T$; for $i = m+1, m+2, \ldots$, there are at most two dyadic intervals of length $2^{-i}T$ (start from finding the point in $[s,t]$ of the form $2^{-k}T$ with the smallest possible k and cover $(s, 2^{-k}T]$ and $[2^{-k}T, t)$ by dyadic intervals in the greedy manner). Combining (12) and $2^{-m}T \le \delta$, we obtain:

$$\begin{aligned}
m_\delta^T(\omega) &\le 2 \times 2^{3/2} \left(2^{1/4}-1\right)^{-1/2} \alpha^{-1/2} T^{1/2} \\
&\quad \times \left(2^{-m/8} + 2^{-(m+1)/8} + 2^{-(m+2)/8} + \cdots\right) \\
&= 2^{5/2} \left(2^{1/4}-1\right)^{-1/2} \left(1-2^{-1/8}\right)^{-1} \alpha^{-1/2} T^{1/2} 2^{-m/8} \\
&\le 2^{5/2} \left(2^{1/4}-1\right)^{-1/2} \left(1-2^{-1/8}\right)^{-1} \alpha^{-1/2} T^{1/2} (\delta/T)^{1/8} \\
&= 2^{5/2} \left(2^{1/4}-1\right)^{-1/2} \left(1-2^{-1/8}\right)^{-1} \alpha^{-1/2} T^{3/8} \delta^{1/8},
\end{aligned}$$

which is stronger than (10). □

Now we can prove the following elaboration of Lemma 2, which will also be used in the next section.

Lemma 4. *For each $\alpha > 0$,*

$$\underline{\mathbb{P}}' \left\{\forall T \ge 1\ \forall \delta > 0 : m_\delta^T \le 560\, \alpha^{-1/2} T^{1/2} \delta^{1/8}\right\} \ge 1 - \alpha. \tag{13}$$

Proof. Replacing α in (10) by $\alpha_T := (1-2^{-1/4})T^{-1/4}\alpha$ for $T = 1,2,4,8,\ldots$ (where $1-2^{-1/4}$ is the normalizing constant ensuring that the α_T sum to α over T), we obtain

$$\underline{\mathbb{P}}' \left\{\forall \delta > 0 : m_\delta^T \le 157\,(1-2^{-1/4})^{-1/2} \alpha^{-1/2} T^{1/2} \delta^{1/8}\right\} \ge 1-(1-2^{-1/4})T^{-1/4}\alpha.$$

The countable subadditivity of upper probability now gives

$$\begin{aligned}
\underline{\mathbb{P}}' \left\{\forall T \in \{1,2,4,\ldots\}\ \forall \delta > 0 : m_\delta^T \le 157\,(1-2^{-1/4})^{-1/2} \alpha^{-1/2} T^{1/2} \delta^{1/8}\right\} \\
\ge 1 - \alpha,
\end{aligned}$$

which in turn gives

$$\underline{\mathbb{P}}' \left\{\forall T \ge 1\ \forall \delta > 0 : m_\delta^T \le 157\,(1-2^{-1/4})^{-1/2} \alpha^{-1/2} (2T)^{1/2} \delta^{1/8}\right\} \ge 1 - \alpha,$$

which is stronger than (13). □

Lemma 2 immediately follows from Lemma 4 and the Arzelà–Ascoli theorem (as stated in [8], Theorem 2.4.9).

Inequality (11) will also be useful in the next section; the following lemma packages it in a convenient form.

Lemma 5. *For each $\alpha > 0$,*

$$\underline{\mathbb{P}}'\left\{\forall T \in \{1,2,4,\dots\}\ \forall n \in \{1,2,\dots\} : \sum_{i=1}^{2^n}\Big(\omega(i2^{-n}T) - \omega((i-1)2^{-n}T)\Big)^2 \right. \\ \left. \le 46\,\alpha^{-1}T^2 2^{n/16}\right\} \ge 1-\alpha. \quad (14)$$

Proof. Replacing $\beta/2$ in (11) with $2^{-1}(2^{1/16}-1)T^{-1}2^{-n/16}\alpha$, where T ranges over $\{1,2,4,\dots\}$ and n over $\{1,2,\dots\}$, we obtain

$$\underline{\mathbb{P}}'\left\{\sum_{i=1}^{2^n}\Big(\omega(i2^{-n}T) - \omega((i-1)2^{-n}T)\Big)^2 \right. \\ \left. \le 2(2^{1/16}-1)^{-1}\alpha^{-1}T^2 2^{n/16}\right\} \ge 1 - 2^{-1}(2^{1/16}-1)T^{-1}2^{-n/16}\alpha;$$

by the countable subadditivity of upper probability this implies

$$\underline{\mathbb{P}}'\left\{\forall T \in \{1,2,4,\dots\}\ \forall n \in \{1,2,\dots\} : \sum_{i=1}^{2^n}\Big(\omega(i2^{-n}T) - \omega((i-1)2^{-n}T)\Big)^2 \right. \\ \left. \le 2(2^{1/16}-1)^{-1}\alpha^{-1}T^2 2^{n/16}\right\} \ge 1-\alpha,$$

which is stronger than (14). ∎

8 Proof of (7) for the modified Lévy game

To establish (7) (with $\overline{\mathbb{E}}$ replaced by $\overline{\mathbb{E}}'$) we only need to establish $\overline{\mathbb{E}}'(F) < \int F dW + \epsilon$ for a positive constant ϵ. We start from a series of reductions:

1. We can assume that F is lower semicontinuous on Ω. Indeed, if it is not, by the Vitali–Carathéodory theorem (see, e.g., [13], Theorem 2.24) for any compact $K \subseteq \Omega$ there exists a lower semicontinuous function G on K such that $G \ge F$ on K and $\int_K G dW \le \int_K F dW + \epsilon$. Without loss of generality we assume $\sup G \le \sup F$, and we extend G to all of Ω by setting $G := \sup F$ outside K. Choosing K with large enough $W(K)$ (which can be done since the probability measure W is tight: see, e.g., [2], Theorem 1.4), we will have $G \ge F$ and $\int G dW \le \int F dW + 2\epsilon$. Achieving $S_0 \le \int G dW + \epsilon$ and $\liminf_{t\to\infty} S_t(\omega) \ge G(\omega)$, where S is a capital process, will automatically achieve $S_0 \le \int F dW + 3\epsilon$ and $\liminf_{t\to\infty} S_t(\omega) \ge F(\omega)$.

2. We can further assume that F is continuous on Ω. Indeed, since each lower semicontinuous function on a metric space is a limit of an increasing sequence of continuous functions (see, e.g., [5], Problem 1.7.15(c)), given a lower semicontinuous function F on Ω we can find a series of positive continuous functions G^n on Ω, $n = 1, 2, \ldots$, such that $\inf F + \sum_{n=1}^{\infty} G^n = F$. The sum S of $\inf F$ and positive capital processes $S^1, S^2, \ldots$ achieving $S_0^n \le \int G^n dW + 2^{-n}\epsilon$ and $\liminf_{t\to\infty} S_t^n(\omega) \ge G^n(\omega)$, $n = 1, 2, \ldots$, will achieve $S_0 \le \int F dW + \epsilon$ and $\liminf_{t\to\infty} S_t(\omega) \ge F(\omega)$.

3. We can further assume that F depends on $\omega \in \Omega$ only via $\omega|_{[0,T]}$ for some $T \in (0, \infty)$. Indeed, let us fix $\epsilon > 0$ and prove $\overline{\mathbb{E}}'(F) \le \int F dW + C\epsilon$ for some positive constant C assuming $\overline{\mathbb{E}}'(G) \le \int G dW$ for all G that depend on ω only via $\omega|_{[0,T]}$ for some $T \in (0, \infty)$. Choose a compact set $K \subseteq \Omega$ with $W(K) > 1 - \epsilon$ and $\underline{\mathbb{P}}'(K) > 1 - \epsilon$ (cf. Lemma 2). Set $F^T(\omega) := F(\omega^T)$, where ω^T is defined by $\omega^T(t) := \omega(t \wedge T)$ and T is sufficiently large in the following sense. Since F is uniformly continuous on K and the metric is defined by (9), F and F^T can be made arbitrarily close in $C(K)$ (spaces $C(\ldots)$ are always equipped with the sup norm in this paper); in particular, let $\|F - F^T\|_{C(K)} < \epsilon$. Choose capital processes S^0 and S^1 such that $S_0^0 \le \int F^T dW + \epsilon$, $\liminf_{t\to\infty} S_t^0(\omega) \ge F^T(\omega)$, $S_0^1 \le \epsilon$, $\liminf_{t\to\infty} S_t^1(\omega) \ge \mathbb{I}_{K^c}(\omega)$. The sum $S := S^0 + 2\sup|F| S^1 + \epsilon$ will satisfy

$$\begin{aligned} S_0 &\le \int F^T dW + (2\sup|F| + 2)\epsilon \le \int_K F^T dW + (3\sup|F| + 2)\epsilon \\ &\le \int_K F dW + (3\sup|F| + 3)\epsilon \le \int F dW + (4\sup|F| + 3)\epsilon \end{aligned}$$

and

$$\liminf_{t\to\infty} S_t(\omega) \ge F^T(\omega) + 2\sup|F|\,\mathbb{I}_{K^c}(\omega) + \epsilon \ge F(\omega).$$

Without loss of generality, we assume $T \in \{1, 2, 4, \ldots\}$.

4. We can further assume that F depends on ω only via the values $\omega(iT/N)$, $i = 1, \ldots, N$ (remember that $\omega(0) = 0$), for some $N \in \{1, 2, \ldots\}$. Indeed, let us fix $\epsilon > 0$ and prove $\overline{\mathbb{E}}'(F) \le \int F dW + C\epsilon$ for some positive constant C assuming $\overline{\mathbb{E}}'(G) \le \int G dW$ for all G that depend on ω only via $\omega(iT/N)$, $i = 1, \ldots, N$, for some N. Let $K \subseteq \Omega$ be the compact set in Ω defined as $K := \{\omega \mid \forall\delta > 0 : m_\delta^T \le f(\delta)\}$ for some $f : (0, \infty) \to (0, \infty)$ satisfying $\lim_{\delta\to 0} f(\delta) = 0$ (cf. the Arzelà–Ascoli theorem) and chosen in such a way that $W(K) > 1 - \epsilon$ and $\underline{\mathbb{P}}'(K) > 1 - \epsilon$. Let g be a modulus of continuity of F on K; we know that $\lim_{\delta\to 0} g(\delta) = 0$. Set $F_N(\omega) := F(\omega_N)$, where ω_N is the piecewise linear function whose graph is obtained by joining the points $(iT/N, \omega(iT/N))$, $i = 0, 1, \ldots, N$, and $(\infty, \omega(T))$, and N is so large that $g(f(T/N)) \le \epsilon$. Since

$$\omega \in K \implies \|\omega - \omega_N\|_{C([0,T])} \le f(T/N) \implies \rho(\omega, \omega_N) \le f(T/N)$$

(we assume, without loss of generality, that the graph of ω is horizontal over $[T, \infty)$), we have $\|F - F_N\|_{C(K)} \le \epsilon$. Choose capital processes S^0

and S^1 such that $S^0_0 \le \int F_N dW + \epsilon$, $\liminf_{t\to\infty} S^0_t(\omega) \ge F_N(\omega)$, $S^1_0 \le \epsilon$, $\liminf_{t\to\infty} S^1_t(\omega) \ge \mathbb{I}_{K^c}(\omega)$. The sum $S := S^0 + 2\sup|F| S^1 + \epsilon$ will satisfy

$$\begin{aligned} S_0 &\le \int F_N dW + (2\sup|F| + 2)\epsilon \le \int_K F_N dW + (3\sup|F| + 2)\epsilon \\ &\le \int_K F dW + (3\sup|F| + 3)\epsilon \le \int F dW + (4\sup|F| + 3)\epsilon \end{aligned}$$

and

$$\liminf_{t\to\infty} S_t(\omega) \ge F_N(\omega) + 2\sup|F|\,\mathbb{I}_{K^c}(\omega) + \epsilon \ge F(\omega).$$

5. We can further assume that

$$F(\omega) = U\left(\omega(T/N), \omega(2T/N), \dots, \omega(T)\right) \tag{15}$$

where the function $U : \mathbb{R}^N \to \mathbb{R}$ is not only continuous but also has compact support. (We will sometimes say that U is the *generator* of F.) Indeed, let us fix $\epsilon > 0$ and prove $\overline{\mathbb{E}}'(F) \le \int F dW + C\epsilon$ for some positive constant C assuming $\overline{\mathbb{E}}'(G) \le \int G dW$ for all G whose generator has compact support. Let B_R be the open ball of radius R and centred at the origin in the space $\mathbb{R}^N$ with the ℓ_∞ norm. We can rewrite (15) as $F(\omega) = U(s(\omega))$ where $s : \Omega \to \mathbb{R}^N$ reduces each $\omega \in \Omega$ to $s(\omega) := (\omega(T/N), \omega(2T/N), \dots, \omega(T))$. Choose R so large that $W(s^{-1}(B_R)) > 1 - \epsilon$ and $\underline{\mathbb{P}}'(s^{-1}(B_R)) > 1 - \epsilon$ (the existence of such R follows from the Arzelà–Ascoli theorem and Lemma 2). Alongside F, whose generator is denoted U, we will also consider F^* with generator

$$U^*(\sigma) := \begin{cases} U(\sigma) & \text{if } \sigma \in \overline{B_R} \\ 0 & \text{if } \sigma \in B^c_{2R}; \end{cases}$$

in the remaining region $B_{2R} \setminus \overline{B_R}$, U^* is defined arbitrarily (but making sure that U^* is continuous and takes values in $[\inf U, \sup U]$; this can be done by the Tietze–Urysohn theorem, [5], Theorem 2.1.8). Choose capital processes S^0 and S^1 such that $S^0_0 \le \int F^* dW + \epsilon$, $\liminf_{t\to\infty} S^0_t(\omega) \ge F^*(\omega)$, $S^1_0 \le \epsilon$, $\liminf_{t\to\infty} S^1_t(\omega) \ge \mathbb{I}_{s^{-1}(B^c_R)}(\omega)$. The sum $S := S^0 + 2\sup|F| S^1$ will satisfy

$$\begin{aligned} S_0 &\le \int F^* dW + (2\sup|F| + 1)\epsilon \le \int_{s^{-1}(B_R)} F^* dW + (3\sup|F| + 1)\epsilon \\ &= \int_{s^{-1}(B_R)} F dW + (3\sup|F| + 1)\epsilon \le \int F dW + (4\sup|F| + 1)\epsilon \end{aligned}$$

and

$$\liminf_{t\to\infty} S_t(\omega) \ge F^*(\omega) + 2\sup|F|\,\mathbb{I}_{s^{-1}(B^c_R)}(\omega) \ge F(\omega).$$

6. Since every continuous $U : \mathbb{R}^N \to \mathbb{R}$ with compact support can be arbitrarily well approximated in $C(\mathbb{R}^N)$ by an infinitely differentiable function with compact support (see, e.g., [1], Theorem 2.29(d)), we can further assume that the generator U of F is an infinitely differentiable function with compact support.

7. By Lemma 2, it suffices to prove that, given $\epsilon > 0$ and a compact set K in Ω, some capital process $S \geq \inf F - 1$ with $S_0 \leq \int F dW + \epsilon$ achieves $\liminf_{t\to\infty} S_t(\omega) \geq F(\omega)$ for all $\omega \in K$. Indeed, we can choose K with $\underline{\mathbb{P}}'(K)$ so close to 1 that the sum of S and a positive capital process eventually attaining $2\sup|F|+2$ on K^c will give a capital process starting from $\int F dW + 2\epsilon$ and exceeding $F(\omega)$ in the limit.

From now on we fix a compact $K \subseteq \Omega$, assuming, without loss of generality, that the statements inside the curly braces in (13) and (14) are satisfied for some $\alpha > 0$.

In the rest of the proof we will be using, often following [14], Section 6.2, the standard method going back to Lindeberg [12]. For $i = N-1$, define a function $\overline{U}_i : \mathbb{R} \times [0,\infty) \times \mathbb{R}^i \to \mathbb{R}$ by

$$\overline{U}_i(s, D; s_1, \dots, s_i) := \int_{-\infty}^{\infty} U_{i+1}(s_1, \dots, s_i, s+z)\mathcal{N}_{0,D}(dz), \tag{16}$$

where U_N stands for U and $\mathcal{N}_{0,D}$ is the Gaussian probability measure on $\mathbb{R}$ with mean 0 and variance $D \geq 0$. Next define, for $i = N-1$,

$$U_i(s_1, \dots, s_i) := \overline{U}_i(s_i, T/N; s_1, \dots, s_i). \tag{17}$$

Finally, we can alternately use (16) and (17) for $i = N-2, \dots, 1, 0$ to define inductively other $\overline{U}_i$ and U_i (with (17) interpreted as $U_0 := \overline{U}_0(0, T/N)$ when $i = 0$). Notice that $U_0 = \int F dW$.

Informally, the functions (16) and (17) constitute Sceptic's goal: assuming $\omega \in K$, he will keep his capital at time iT/N, $i = 0, 1, \dots, N$, close to $U_i(\omega(T/N), \omega(2T/N), \dots, \omega(iT/N))$ and his capital at any other time $t \in [0,T]$ close to $\overline{U}_i(\omega(t), D; \omega(T/N), \omega(2T/N), \dots, \omega(iT/N))$ where $i := \lfloor Nt/T \rfloor$ and $D := (i+1)T/N - t$. This will ensure that his capital at time T is close to $F(\omega)$ when his initial capital is $U_0 = \int F dW$.

It is easy to check that each function $\overline{U}_i(s, D; s_1, \dots, s_i)$ satisfies the heat equation in the variables s and D:

$$\frac{\partial \overline{U}_i}{\partial D}(s, D; s_1, \dots, s_i) = \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial s^2}(s, D; s_1, \dots, s_i) \tag{18}$$

for all $s \in \mathbb{R}$, all $D > 0$, and all $s_1, \dots, s_i \in \mathbb{R}$. This is the key element of the proof.

Sceptic will only bet at the times that are multiples of T/LN, where $L \in \{1, 2, \dots\}$ will later be chosen large. For $i = 0, \dots, N$ and $j = 0, \dots, L$ let us set

$$t_{i,j} := iT/N + jT/LN, \quad S_{i,j} := \omega(t_{i,j}), \quad D_{i,j} := T/N - jT/LN.$$

For any array $A_{i,j}$, we set $dA_{i,j} := A_{i,j+1} - A_{i,j}$.

Using Taylor's formula and omitting the arguments $\omega(T/N), \ldots, \omega(iT/N)$, we obtain, for $i = 0, \ldots, N-1$ and $j = 0, \ldots, L-1$,

$$\begin{aligned} d\overline{U}_i(S_{i,j}, D_{i,j}) = \frac{\partial \overline{U}_i}{\partial s}(S_{i,j}, D_{i,j}) dS_{i,j} + \frac{\partial \overline{U}_i}{\partial D}(S_{i,j}, D_{i,j}) dD_{i,j} \\ + \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial s^2}(S'_{i,j}, D'_{i,j})(dS_{i,j})^2 + \frac{\partial^2 \overline{U}_i}{\partial s \partial D}(S'_{i,j}, D'_{i,j}) dS_{i,j} dD_{i,j} \\ + \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial D^2}(S'_{i,j}, D'_{i,j})(dD_{i,j})^2, \end{aligned} \tag{19}$$

where $(S'_{i,j}, D'_{i,j})$ is a point strictly between $(S_{i,j}, D_{i,j})$ and $(S_{i,j+1}, D_{i,j+1})$. Applying Taylor's formula to $\partial^2 \overline{U}_i / \partial s^2$, we find

$$\begin{aligned} \frac{\partial^2 \overline{U}_i}{\partial s^2}(S'_{i,j}, D'_{i,j}) = \frac{\partial^2 \overline{U}_i}{\partial s^2}(S_{i,j}, D_{i,j}) \\ + \frac{\partial^3 \overline{U}_i}{\partial s^3}(S''_{i,j}, D''_{i,j}) \Delta S_{i,j} + \frac{\partial^3 \overline{U}_i}{\partial D \partial s^2}(S''_{i,j}, D''_{i,j}) \Delta D_{i,j}, \end{aligned}$$

where $(S''_{i,j}, D''_{i,j})$ is a point strictly between $(S_{i,j}, D_{i,j})$ and $(S'_{i,j}, D'_{i,j})$, and $\Delta S_{i,j}$ and $\Delta D_{i,j}$ satisfy $|\Delta S_{i,j}| \le |dS_{i,j}|$, $|\Delta D_{i,j}| \le |dD_{i,j}|$. Plugging this equation and the heat equation (18) into (19), we obtain

$$\begin{aligned} d\overline{U}_i(S_{i,j}, D_{i,j}) = \frac{\partial \overline{U}_i}{\partial s}(S_{i,j}, D_{i,j}) dS_{i,j} + \frac{1}{2}\frac{\partial^2 \overline{U}_i}{\partial s^2}(S_{i,j}, D_{i,j}) \left((dS_{i,j})^2 + dD_{i,j}\right) \\ + \frac{1}{2}\frac{\partial^3 \overline{U}_i}{\partial s^3}(S''_{i,j}, D''_{i,j}) \Delta S_{i,j} (dS_{i,j})^2 + \frac{1}{2}\frac{\partial^3 \overline{U}_i}{\partial D \partial s^2}(S''_{i,j}, D''_{i,j}) \Delta D_{i,j} (dS_{i,j})^2 \\ + \frac{\partial^2 \overline{U}}{\partial s \partial D}(S'_{i,j}, D'_{i,j}) dS_{i,j} dD_{i,j} + \frac{1}{2}\frac{\partial^2 \overline{U}}{\partial D^2}(S'_{i,j}, D'_{i,j})(dD_{i,j})^2. \end{aligned} \tag{20}$$

At this point we are at last ready to define Sceptic's elementary betting strategy: namely, he plays in such a way that the increment of his capital between times $t_{i,j}$ and $t_{i,j+1}$ is equal to the sum of the first two terms on the right-hand side of (20). Both $\partial \overline{U}_i / \partial s$ and $\partial^2 \overline{U}_i / \partial s^2$ are bounded as averages of $\partial U_{i+1} / \partial s$ and $\partial^2 U_{i+1} / \partial s^2$, and so, eventually, averages of $\partial U / \partial s$ and $\partial^2 U / \partial s^2$, respectively.

Let us show that the last four terms on the right-hand side of (20) are negligible when L is sufficiently large (assuming T, N, and U fixed). All the partial derivatives involved in those terms are bounded: the heat equation implies

$$\begin{aligned} \frac{\partial^3 \overline{U}_i}{\partial D \partial s^2} &= \frac{\partial^3 \overline{U}_i}{\partial s^2 \partial D} = \frac{1}{2}\frac{\partial^4 \overline{U}_i}{\partial s^4}, \\ \frac{\partial^2 \overline{U}_i}{\partial s \partial D} &= \frac{1}{2}\frac{\partial^3 \overline{U}_i}{\partial s^3}, \\ \frac{\partial^2 \overline{U}_i}{\partial D^2} &= \frac{1}{2}\frac{\partial^3 \overline{U}_i}{\partial D \partial s^2} = \frac{1}{4}\frac{\partial^4 \overline{U}_i}{\partial s^4}, \end{aligned}$$

and $\partial^3\overline{U}_i/\partial s^3$ and $\partial^4\overline{U}_i/\partial s^4$, being averages of $\partial^3 U_{i+1}/\partial s^3$ and $\partial^4 U_{i+1}/\partial s^4$, and eventually averages of $\partial^3 U/\partial s^3$ and $\partial^4 U/\partial s^4$, are bounded. We can assume that

$$|dS_{i,j}| \le C_1 L^{-1/8}, \quad \sum_{i=0}^{N-1}\sum_{j=0}^{L-1}(dS_{i,j})^2 \le C_2 L^{1/16}$$

(cf. (13) and (14), respectively) for $\omega \in K$ and some constants C_1 and C_2 (remember that T, N, U, and, of course, α are fixed; without loss of generality we assume that N and L are powers of 2). This makes the cumulative contribution of the four terms have at most the order of magnitude $O(L^{-1/16})$; therefore, Sceptic can achieve his goal for $\omega \in K$ by making L sufficiently large.

To ensure that his capital never drops strictly below $\inf F - 1$, Sceptic stops playing as soon as his capital hits $\inf F - 1$. This will never happen when $\omega \in K$ (for L sufficiently large).

9 Proofs for the Lévy game

The following simple lemma will allow us to deduce the results for the Lévy game.

Lemma 6. *For each $\alpha > 0$ and $T > 0$, we have*

$$\underline{\mathbb{P}}\left\{\sup_{t\in[0,T]}|\omega(t)| \le \alpha^{-1/2}T^{1/2}\right\} \ge 1-\alpha \qquad (21)$$

in the Lévy game.

Proof. Starting from initial capital α, bet α/T on $\omega^2(t) - t$ at time 0 and stop playing (set the stake to 0) at time $T \wedge \inf\{t \mid |\omega(t)| = \alpha^{-1/2}T^{1/2}\}$; the initial capital α will grow to at least $\alpha + \frac{\alpha}{T}(\alpha^{-1}T - T) = 1$ if the inner inequality in (21) is violated. $\square$

It is instructive to compare the bounds given by the inner inequalities in (10) with $\delta := T$ and in (21): the only difference is the constant factor 157 in the former. Lemma 3 itself will also continue to hold in the Lévy game; we will state it in a slightly weakened form:

Lemma 7. *For each $\alpha > 0$ and $T > 0$,*

$$\underline{\mathbb{P}}\left\{\forall\delta > 0 : m_\delta^T \le 157\,\alpha^{-1/2}T^{3/8}\delta^{1/8}\right\} \ge 1 - 2\alpha.$$

Proof. Let $\tau := \inf\left\{t \ge 0 \mid |\omega(t)| = \alpha^{-1/2}T^{1/2} + 1\right\}$ and let $\omega^\tau : [0,\infty) \to \mathbb{R}$ be the stopped ω, $\omega^\tau(t) := \omega(\tau \wedge t)$. Consider the same elementary betting strategies as in the proof of Lemma 3 except that they now stop playing at time τ. Identity (8) with ω^τ in place of ω shows that the corresponding elementary capital processes will also be elementary capital processes in the Lévy game.

Their combination (analogous to the one in the proof of Lemma 3) witnesses that

$$\sup_{t\in[0,T]} |\omega(t)| \le \alpha^{-1/2}T^{1/2} \implies \forall\delta > 0 : m_\delta^T \le 157\,\alpha^{-1/2}T^{3/8}\delta^{1/8}$$

with lower probability at least $1-\alpha$ in the Lévy game; it remains to combine this with Lemma 6. $\square$

In the same way as we obtained Lemma 4 from Lemma 3, we can now obtain the following corollary from Lemma 7:

Lemma 8. *For each $\alpha > 0$,*

$$\underline{\mathbb{P}}\left\{\forall T \ge 1\ \forall\delta > 0 : m_\delta^T \le 800\,\alpha^{-1/2}T^{1/2}\delta^{1/8}\right\} \ge 1-\alpha.$$

Lemma 8 shows that Lemma 2 will also hold for the Lévy game. In a similar way we can get rid of the prime in $\underline{\mathbb{P}}'$ in Lemma 5.

The proof of (7) for the Lévy game proceeds as in the case of the modified Lévy game: indeed, the 7 reductions in Section 8 do not depend on the game being played, and the elementary betting strategy constructed afterwards always makes bounded stakes (see its description after (20)).

10 Conclusion

In this short section we will state two open problems. First, what is the class $\mathcal{A}$ of all $\overline{\mathbb{P}}$-measurable subsets of Ω? It is easy to see that the statement of Theorem 1 that the sets in $\mathcal{F}$ are $\overline{\mathbb{P}}$-measurable can be strengthened:

Proposition 2. *Each set $A \in \mathcal{F}^W$ in the completion of $\mathcal{F}$ w.r. to W is $\overline{\mathbb{P}}$-measurable.*

Proof. To establish (6) for $A \in \mathcal{F}^W$ we choose $A_1, A_2 \in \mathcal{F}$ such that $A_1 \subseteq A \subseteq A_2$ and $W(A_1) = W(A_2)$, and define F as in the proof of Theorem 1 (see Section 4). Since now $\overline{\mathbb{E}}(F) < \overline{\mathbb{P}}(E) + \epsilon$, $\mathbb{I}_{E\cap A} \le F\,\mathbb{I}_A \le F\,\mathbb{I}_{A_2}$, and $\mathbb{I}_{E\cap A^c} \le F\,\mathbb{I}_{A^c} \le F\,\mathbb{I}_{A_1^c}$, it suffices to notice that

$$\overline{\mathbb{E}}(F\,\mathbb{I}_{A_2}) + \overline{\mathbb{E}}(F\,\mathbb{I}_{A_1^c}) \le \overline{\mathbb{E}}(F)$$

immediately follows from Theorem 2. $\square$

In particular, it would be interesting to know whether $\mathcal{A}$ coincides with $\mathcal{F}^W$.

The second problem is: will the (modified) Lévy game remain coherent if the measurability restrictions on stopping times and stakes are dropped? (In other words, if each σ-algebra considered is extended to become closed under arbitrary, and not just countable, unions and intersections.) A positive answer would lead to simpler and more intuitive definitions. A negative answer would also be of great interest, providing a counter-intuitive phenomenon akin to the Banach–Tarski paradox. A related question is whether dropping the requirement that M and V should be bounded will lead to loss of coherence.

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